

# Definite Descriptions in KR Languages

## Half-day Tutorial

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# Outline

- 1 Structure of the Tutorial

# Structure of the Tutorial

## ① Structure of the Tutorial

### PART 1 Introduction to the topics

- ① Non-denoting terms and free logics  $\leadsto$  **Oliver**
- ② Definite descriptions and proofs  $\leadsto$  **Andrzej**

### PART 2 Free descriptions logics with definite descriptions $\leadsto$ **Przemek**

### PART 3 First-order modal and temporal logics with definite descriptions $\leadsto$ **Alessandro, Andrea**

To be or not to be ...



# Introduction: Use Cases for the non-existing

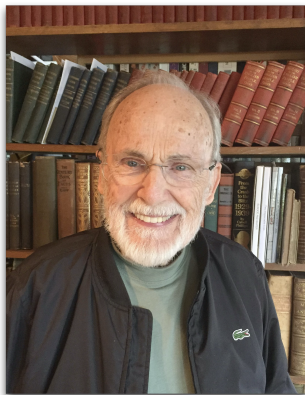
② To be or not to be ...

③ Introduction

# Free Logics: Reasoning with the non-existing

② To be or not to be ...

③ Introduction



**Karel Lambert (1960)** coined the term '**free logic**' as an abbreviation for 'logic free of existence assumptions with respect to its terms, singular and general'.

## Existential presupposition: Classical logic

- Each singular term  $t$  **refers** to something in the domain of interpretation:

$$\exists x(x = t)$$

- There **exists** something in the domain of interpretation (i.e., it is non-empty)

↪ These assumptions are **inherited** by Description Logic languages.

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- Existence is treated as a **predicate** (primitive or defined):

$$E!(t) =_{def} \exists x(x = t)$$

Now we can say ‘*t* does not exist’:  $\neg E!(t)$ .

**Quantification must range** over all and only the objects that satisfy the existence predicate.

Why do we need Free Logics?  
Examples of non-existing entities

Fictional entities: The Hulk (is green) and the Unicorn (is pink)



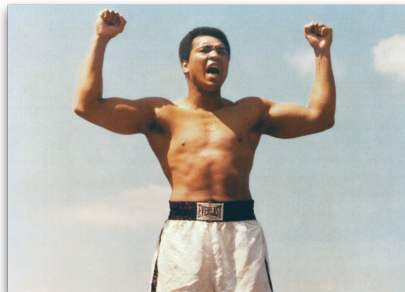
Fantastic Four Annual vol. 1, 1  
(July 1963), Wikipedia.



'Unicorn', screenshot from <https://www.cornify.com/>



## Fictional versus real entities: The Hulk and The Greatest



Consider

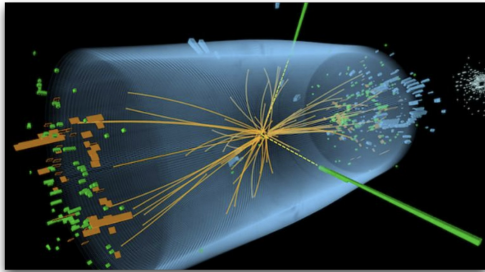
*“The Hulk is green” and “Muhammad Ali is The Greatest”*

versus

*“The Hulk is stronger than Muhammad Ali”*

Do we want to allow only **positive concept attributions** directly about non-existing objects, or also **relations between** the existing and the non-existing?

# Postulated entities: The Higgs boson, came into existence July 4, 2012



July 4, 2012

The New York Times

*Physicists Find Elusive Particle Seen  
as Key to Universe*

Search - The Guardian International edition -

The Higgs boson discovery is another giant leap for humankind

The Higgs boson [...] is an elementary particle in the Standard Model of particle physics produced by the quantum excitation of the Higgs field, one of the fields in particle physics theory. [...] After a 40-year search, a subatomic particle with the expected properties was discovered in 2012 by the ATLAS and CMS experiments at the Large Hadron Collider (LHC) at CERN near Geneva, Switzerland. (Source: Wikipedia, July 2026)

# Idealisations: The ideal gas

## Ideal Gas Law

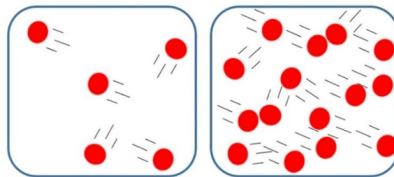
Diagram illustrating the Ideal Gas Law equation:  $PV = Nk_B T$ .

Labels and arrows indicating the variables:

- Pressure** (green box) points to  $P$ .
- Volume** (blue box) points to  $V$ .
- Number of particles** (white box) points to  $N$ .
- Boltzman's constant** (white box) points to  $k_B$ .
- Temperature** (orange box) points to  $T$ .

Value of Boltzman's constant:  $k_B = 1.38 \times 10^{-23} \text{ m}^2 \text{ kg s}^{-2} \text{ K}^{-1}$

In words: The product of **pressure** and **volume** is equal to the **number of particles** multiplied by the **Boltzman's constant** and **temperature**.



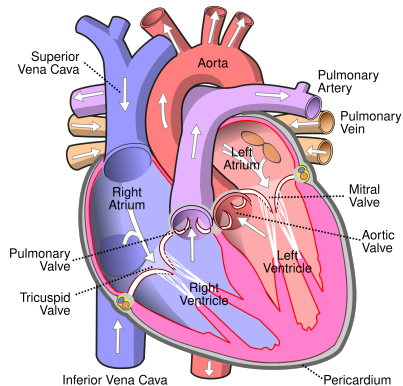
Ideal Gas

Real Gas

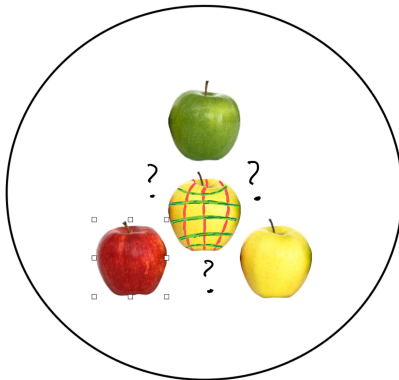
An ideal gas is a theoretical gas composed of many randomly moving point particles that are not subject to interparticle interactions. The ideal gas concept is useful because it obeys the ideal gas law, a simplified equation of state, and is amenable to analysis under statistical mechanics.

(Source: Wikipedia, July 2026)

# Idealisations: Canonical hearts / Prototypes



A perfect 'textbook heart' as, in reality, it does not exist.



A 'good' object realising all salient properties might not exist.

# Errors, glitches, and retractions

## Errors in the modelling process

The screenshot shows a browser window with the URL `http://www.semanticweb.org/ontologies/example#t56792048`. The page displays annotations for `t56792048`. A red error message is visible in the console, stating: `"error": "error querying iri=http://sweetontology.net/stateSystem/Artificial: java.net.ConnectException: "`. Below the error, the URL `http://sweetontology.net/stateSystem/Artificial` is highlighted, and a message indicates `IRI not found` with the reasons: 

- No ontology registered with this IRI
- No subject in the triple store found with this IRI

Objects can become 'obsolete',  
names (IRIs) might point nowhere.

The screenshot shows the arXiv website for a paper titled "Your Paper's Title" by "Your Name". The paper is marked as "Withdrawn". A yellow circle highlights the text "This paper has been withdrawn by Your Name". Another yellow circle highlights the "Download:" section, which includes a "Withdrawn" status. A third yellow circle highlights the "Comments" section, which states: "The explanation for the withdrawal appears here in the Comments". A fourth yellow circle highlights the "Submission history" section, which shows the paper was submitted on May 12, 2023, and is currently "withdrawn".

Papers can go out of existence, and referencing them can fail,  
and links can die. At a later point, they can come back into  
existence (new version).

Free Logics come in families

Semantic choices  
for non-referring terms

## Is “The Hulk is green” true or false?

Three different approaches to the semantics.

Consider a statement of the form  $C(a)$  where  $a$  does **not** refer

- It is *TRUE*: **Positive Free Logic**

$\leadsto$  e.g. modelling fictional information

- It is *FALSE*: **Negative Free Logic**

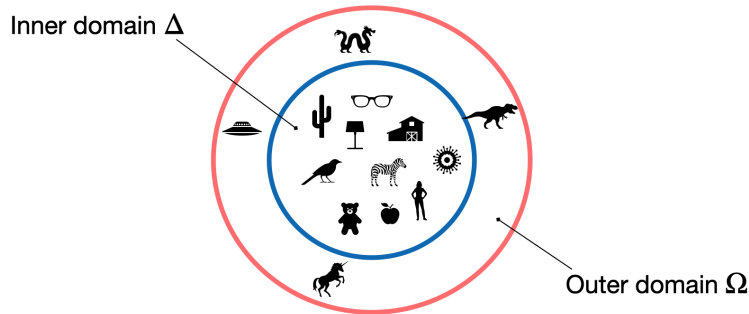
$\leadsto$  e.g. modelling errors

- It is *NEITHER* (true nor false): **Gapped Free Logic** (or neutral, Fregean position)

$\leadsto$  e.g. modelling hypotheses

## Dual domain semantics: basic idea

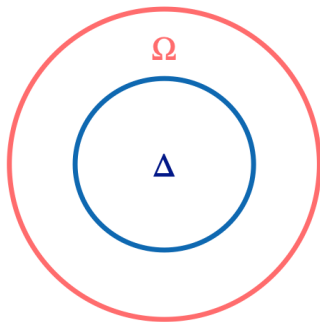
**denoting** terms  $\neq$  **referring** terms



- The inner domain comprises the domain of quantification.
- The outer domain is used to denote terms that are not quantified over.
- Here, **reference implies existence**, denotation does not.



## Dual domain semantics for DLs: Example of $\mathcal{ALCO}$



$$(\perp)^I = \emptyset \quad (T)^I = \Delta^I \quad (\{a\})^I = \{a^I\} \subseteq \Omega$$

$$(\neg A)^I = \Omega \setminus A^I \quad (A \sqcap B)^I = A^I \cap B^I \quad (A \sqcup B)^I = A^I \cup B^I$$

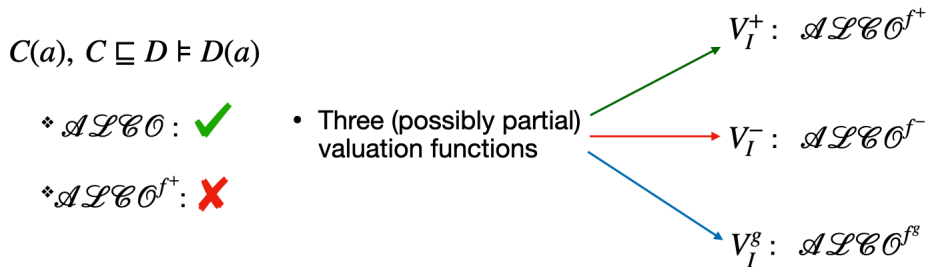
$$(\forall R. C)^I = \{d \in \Omega \mid \text{for all } e \in \Delta^I : (d, e) \in R^I \text{ implies } e \in C^I\}$$

$$(\exists R. C)^I = \{d \in \Omega \mid \text{exists } e \in \Delta^I : (d, e) \in R^I \text{ and } e \in C^I\}$$

$$(A \sqsubseteq B)^I \iff \text{for all } e \in \Delta^I, \text{ if } e \in A^I \text{ then } e \in B^I$$

- **Inner domain** comprises the domain of quantification.
- **Specific choices:** names denote arbitrary objects; subsumptions quantify over the inner domain; quantifiers are over the inner domain, but select from the outer.

# Three basic choices for semantics



- $V_I^+(Cat(tom)) = \mathbf{True} \iff tom^l \in Cat^l$ , and **False** otherwise
- $V_I^-(Cat(tom)) = \mathbf{True} \iff tom^l \in Cat^l$  and  $tom^l \in \Delta$ , and **False** otherwise
- $V_I^g(Cat(tom))$   
 $= \begin{cases} \mathbf{True} & \iff tom^l \in Cat^l \text{ and } tom^l \in \Delta \\ \mathbf{False} & \iff tom^l \notin Cat^l \text{ and } tom^l \in \Delta \\ \text{is undefined,} & \text{otherwise} \end{cases}$

## Example: Bugs Bunny can speak

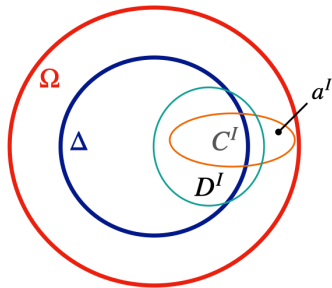
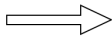
$$C(a), C \sqsubseteq D \models D(a)$$

\*  $ALCO$ : ✓

\*  $ALCO^{f+}$ : ✗



e.g. Bugs Bunny is a rabbit,  
rabbits do not speak, however  
Bugs Bunny speaks.



19

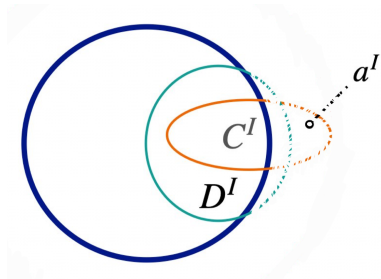
- Since  $a$  is interpreted in the outer domain, it is not subject to the subsumption axiom.
- In positive dual-domain semantics:  $I \models_+^{dd} C(a), \neg D(a), C \sqsubseteq D$  can be all true
- In negative dd semantics:  $I \models_-^{dd} C \sqsubseteq D$  but  $I \not\models_-^{dd} C(a)$  and  $I \not\models_-^{dd} \neg D(a), D(a)$

# Overview of behaviour of free logics

**Table:** The behaviour of some basic  $\mathcal{ALCO}$  assertions under the different semantics. ‘—’ represents that the situation cannot happen.  $\downarrow$  is used for an undefined truth-value.

| Condition on $I$                           | $\mathcal{ALCO}$                                          | $\mathcal{ALCO}^{f+}$ | $\mathcal{ALCO}^{f-}$ | $\mathcal{ALCO}^{fg}$ |
|--------------------------------------------|-----------------------------------------------------------|-----------------------|-----------------------|-----------------------|
| $t' \in \Delta \setminus t' \notin \Delta$ | $I \models \top(t)$                                       | True \ —              | True \ False          | True \ False          |
| $t' \in \Delta \setminus t' \notin \Delta$ | $I \models \neg \top(t)$                                  | False \ —             | False \ True          | False \ False         |
| $t' \in \Delta \setminus t' \notin \Delta$ | $I \models \sim \top(t)$                                  | False \ —             | False \ True          | False \ True          |
| for all $I$                                | $(\neg \top)^I = (\perp)^I$                               | yes                   | no                    | no                    |
|                                            | $\models \neg \top \sqsubseteq \perp$                     | yes                   | yes                   | yes                   |
|                                            | $\models \exists r. C \equiv \neg \forall r. \neg C$      | yes                   | yes                   | yes                   |
|                                            | $C(a), C \sqsubseteq D \models D(a)$                      | yes                   | no                    | yes                   |
|                                            | $C(a) \models \top(a)$                                    | yes                   | no                    | yes                   |
|                                            | $\neg C(a) \models \top(a)$                               | yes                   | no                    | yes                   |
|                                            | $\sim C(a) \models \top(a)$                               | yes                   | no                    | yes                   |
|                                            | $\neg \top(a) \models \exists R. \{a\} \sqsubseteq \perp$ | yes                   | yes                   | yes                   |
|                                            | $R(a, b) \models (\exists R. \{b\})(a)$                   | yes                   | no                    | yes                   |
|                                            | $\models a = a$                                           | yes                   | yes                   | no                    |
|                                            | $\top(a) \models a = a$                                   | yes                   | yes                   | yes                   |

## Equivalent single domain semantics: using partial interpretations



Objects from the outer domain  
are now seen as *non-denoting*

- Interpretations are total on concept and role names
- They are partial on individual names
- Statements about non-denoting terms are false in negative semantics
- We obtain equivalently:  $I \models_{-}^s C \sqsubseteq D$  but  $I \not\models_{-}^s C(a)$  and  $I \not\models_{-}^s \neg D(a), D(a)$
- For positive semantics we can use an existence predicate to simulate the inner domain.

# Reduce dual domain positive/negative semantics to single domain negative semantics with partial interpretations

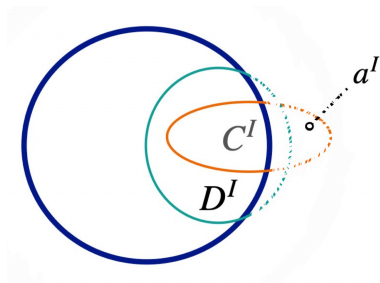
Single-domain negative semantics for  $\mathcal{ALCO}$ :

## Definition (Partial interpretation)

$I = (\Delta^I, \cdot^I)$ , with  $\Delta^I \neq \emptyset$  (domain of  $I$ ), and  $\cdot^I$  as usual, except:

only individual names in a **subset of individual names**  
are mapped to elements of the domain

$$\{\tau\}^I = \begin{cases} \{\tau^I\}, & \text{if } \tau^I \in \Delta^I \\ \emptyset, & \text{otherwise} \end{cases}$$

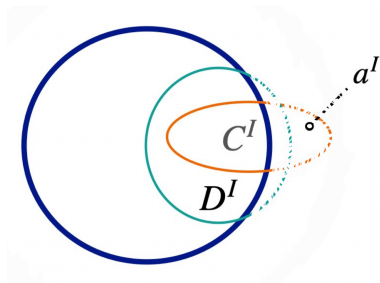


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In the example

- $a^I$  is undefined, thus  $a^I \notin \Delta$
- Thus:  $\{a\} = \emptyset$
- Therefore:  $I \not\models_-^s a : \{a\}$

# Reduce dual domain positive/negative semantics to single domain negative semantics with partial interpretations



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## Theorem

*$\mathcal{ALCO}$  formula satisfiability on dual domain interpretations under either positive or negative semantics is polynomial time reducible to  $\mathcal{ALCO}_u$  ontology satisfiability*

Because of this reduction (which extends to languages including definite descriptions) we will in the following

- develop the technical results systematically for the single-domain negative semantics, and
- consider that these results can be easily (polynomially) transferred to the dual-domain positive/negative scenario.

# Definite Descriptions



# Definite Descriptions: Describing the unique

## ⑤ Definite Descriptions

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Definite descriptions represented by means of iota-operator

$\iota x \varphi(x)$  – the (unique) object  $x$  being  $\varphi$

$\psi(\iota x \varphi(x))$  – The only  $x$  being  $\varphi$  is  $\psi$ .

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- ② shorter proofs of valid cases;
- ③ avoiding infinite models in invalid cases.

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One may avoid troubles (with this example) introducing functional terms (standard solution). But DD are more expressive and can outrun functional terms in many ways.

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## A variety of formal approaches to DD:

- Eliminating approach – Russell;
- Descriptions as genuine terms – Frege:
  - Classical Logic: weak Frege, elimination of improper descriptions  $\Rightarrow$  Hilbert and Bernays, Śłupecki and Borkowski, Stenlund;
  - Classical Logic: strong Frege, fixing denotation for improper descriptions  $\Rightarrow$  Quine, Carnap, Rosser, Bernays, Kalish and Montague;
  - Free Logics: Hintikka, Lambert, Scott, van Fraassen;
  - Partial Logics: Strawson, Blamey;
  - Modal Logics: Carnap, Thomason, Fitting and Mendelsohn, Garson, Goldblatt.
- Definite descriptions as binary quantifiers - Kürbis.

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Show that definite descriptions are not genuine names by means of contextual definitions.



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# Proof-theoretic Perspectives

# Reasoning with definite descriptions and empty terms

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## Russell revisited:

How to develop the Russellian approach in the way avoiding at least some problems while treating DD as genuine terms?

Two solutions:

- Kalish, Montague and Mar [1980] (also Burge [1974], Feferman [1995])  $\implies$  Indrzejczak [RSL 2022]
- use lambda-operator  $\implies$  [Indrzejczak 2020, Indrzejczak and Zawidzki 2022, Indrzejczak and Kürbis 2023]



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Note that if equalities are treated as atomic then RA implies:

(DP-1)  $R^n t_1 \dots t_n \vdash \exists x x = t_1 \wedge \dots \wedge \exists x x = t_n$

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which makes KMM variant of Russellian logic of DD close to negative free logic.

# SEQUENT CALCULUS GRDD

$$(Cut) \frac{\Gamma \Rightarrow \Delta, \varphi \quad \varphi, \Pi \Rightarrow \Sigma}{\Gamma, \Pi \Rightarrow \Delta, \Sigma}$$

$$(AX) \varphi, \Gamma \Rightarrow \Delta, \varphi$$

$$(\neg \Rightarrow) \frac{\Gamma \Rightarrow \Delta, \varphi}{\neg \varphi, \Gamma \Rightarrow \Delta}$$

$$(\Rightarrow \neg) \frac{\varphi, \Gamma \Rightarrow \Delta}{\Gamma \Rightarrow \Delta, \neg \varphi}$$

$$(W \Rightarrow) \frac{\Gamma \Rightarrow \Delta}{\varphi, \Gamma \Rightarrow \Delta}$$

$$(\Rightarrow \wedge) \frac{\Gamma \Rightarrow \Delta, \varphi \quad \Gamma \Rightarrow \Delta, \psi}{\Gamma \Rightarrow \Delta, \varphi \wedge \psi}$$

$$(\wedge \Rightarrow) \frac{\varphi, \psi, \Gamma \Rightarrow \Delta}{\varphi \wedge \psi, \Gamma \Rightarrow \Delta}$$

$$(\Rightarrow W) \frac{\Gamma \Rightarrow \Delta}{\Gamma \Rightarrow \Delta, \varphi}$$

$$(\vee \Rightarrow) \frac{\varphi, \Gamma \Rightarrow \Delta \quad \psi, \Gamma \Rightarrow \Delta}{\varphi \vee \psi, \Gamma \Rightarrow \Delta}$$

$$(\Rightarrow \vee) \frac{\Gamma \Rightarrow \Delta, \varphi, \psi}{\Gamma \Rightarrow \Delta, \varphi \vee \psi}$$

$$(C \Rightarrow) \frac{\varphi, \varphi, \Gamma \Rightarrow \Delta}{\varphi, \Gamma \Rightarrow \Delta}$$

$$(\rightarrow \Rightarrow) \frac{\Gamma \Rightarrow \Delta, \varphi \quad \psi, \Gamma \Rightarrow \Delta}{\varphi \rightarrow \psi, \Gamma \Rightarrow \Delta}$$

$$(\Rightarrow \rightarrow) \frac{\varphi, \Gamma \Rightarrow \Delta, \psi}{\Gamma \Rightarrow \Delta, \varphi \rightarrow \psi}$$

$$(\Rightarrow C) \frac{\Gamma \Rightarrow \Delta, \varphi, \varphi}{\Gamma \Rightarrow \Delta, \varphi}$$

$$(\leftrightarrow \Rightarrow) \frac{\Gamma \Rightarrow \Delta, \varphi, \psi \quad \varphi, \psi, \Gamma \Rightarrow \Delta}{\varphi \leftrightarrow \psi, \Gamma \Rightarrow \Delta}$$

$$(\forall \Rightarrow) \frac{\varphi[x/b], \Gamma \Rightarrow \Delta}{\forall x \varphi, \Gamma \Rightarrow \Delta}$$

$$(\Rightarrow \exists) \frac{\Gamma \Rightarrow \Delta, \varphi[x/b]}{\Gamma \Rightarrow \Delta, \exists x \varphi}$$

$$(\Rightarrow \leftrightarrow) \frac{\varphi, \Gamma \Rightarrow \Delta, \psi \quad \psi, \Gamma \Rightarrow \Delta, \varphi}{\Gamma \Rightarrow \Delta, \varphi \leftrightarrow \psi}$$

$$(\Rightarrow \forall) \frac{\Gamma \Rightarrow \Delta, \varphi[x/a]}{\Gamma \Rightarrow \Delta, \forall x \varphi}$$

$$(\exists \Rightarrow) \frac{\varphi[x/a], \Gamma \Rightarrow \Delta}{\exists x \varphi, \Gamma \Rightarrow \Delta}$$

where  $a$  is a fresh parameter (eigenvariable), not present in  $\Gamma, \Delta$  and  $\varphi$ .

Note: in  $(\forall \Rightarrow), (\Rightarrow \exists)$  only parameters are admitted as instantiated terms!



# SEQUENT CALCULUS GRDD

$$\begin{array}{l}
 (R) \frac{b = b, \Gamma \Rightarrow \Delta}{\Gamma \Rightarrow \Delta} \quad (E) \frac{b_1 = b_2, \Gamma \Rightarrow \Delta}{c = b_1, c = b_2, \Gamma \Rightarrow \Delta} \quad (L) \frac{\varphi[x/b], \Gamma \Rightarrow \Delta}{c = b, \varphi[x/c], \Gamma \Rightarrow \Delta} \\
 (\imath \Rightarrow) \frac{\Gamma \Rightarrow \Delta, \varphi[x/b], c \approx b \quad c \approx b, \varphi[x/b], \Gamma \Rightarrow \Delta}{c \approx \imath x \varphi, \Gamma \Rightarrow \Delta} \quad (\Rightarrow \imath) \frac{c \approx a, \Gamma \Rightarrow \Delta, \varphi[x/a] \quad \varphi[x/a], \Gamma \Rightarrow \Delta, c \approx a}{\Gamma \Rightarrow \Delta, c \approx \imath x \varphi} \\
 (=\Rightarrow) \frac{a_1 = d_1, a_2 = d_2, a_1 = a_2, \Gamma \Rightarrow \Delta}{d_1 = d_2, \Gamma \Rightarrow \Delta} \quad (\Rightarrow =) \frac{\Gamma \Rightarrow \Delta, b_1 = d_1 \quad \Gamma \Rightarrow \Delta, b_2 = d_2 \quad \Gamma \Rightarrow \Delta, b_1 = b_2}{\Gamma \Rightarrow \Delta, d_1 = d_2} \\
 (Str \Rightarrow) \frac{a_1 = d_1, \dots, a_n = d_n, \varphi[x_1/a_1 \dots x_n/a_n], \Gamma \Rightarrow \Delta}{\varphi[x_1/d_1 \dots x_n/d_n], \Gamma \Rightarrow \Delta} \\
 (\Rightarrow Str) \frac{\Gamma \Rightarrow \Delta, b_1 = d_1 \quad \dots \quad \Gamma \Rightarrow \Delta, b_n = d_n \quad \Gamma \Rightarrow \Delta, \varphi[x_1/b_1 \dots x_n/b_n]}{\Gamma \Rightarrow \Delta, \varphi[x_1/d_1 \dots x_n/d_n]}
 \end{array}$$

where:  $a, a_1, a_2$  are different eigenvariables;  $\varphi$  in  $(L)$  is a proper atomic formula;  $c \approx t$  in rules for  $\imath$  means that either  $c = t$  or  $t = c$ ;  $\varphi[x_1/d_1 \dots x_n/d_n]$  in  $(Str \Rightarrow)$  and  $(\Rightarrow Str)$  is a proper quasi-atomic formula with exactly  $n \geq 1$  different definite descriptions as arguments and  $\varphi[x_1/b_1 \dots x_n/b_n]$  is a proper atomic formula

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This approach can be applied also to negative free logic with DD  $\Rightarrow$  [Indrzejczak CADE 2025]