Tutorial: Definite Descriptions in KR Languages
Part II: Description Logics with Definite Descriptions

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Outline

1. Basic Description Logics – recap
2. Description Logics with DDs – main observations
3. Computational complexity analysis
4. Expressive power analysis
   - Definition of appropriate bisimulation
   - Checking bisimilarity algorithmically
5. Reasoning in practice – implementation

Basic Description Logic $\mathcal{ALC}$

$\mathcal{ALC}$ concepts are given by:

$$C := A \mid \neg C \mid C \sqcap C \mid C \sqcup C \mid \exists r.\, C \mid \forall r.\, C$$

corresponding to *multi-modal logic formulae*:

$$\varphi := p \mid \neg\varphi \mid \varphi \land \varphi \mid \varphi \lor \varphi \mid \Diamond_r \varphi \mid \Box_r \varphi$$

Example:
building $\sqcap$ $\forall$shortThan.$\neg$building — 'a building not shorter than any other building'

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Example:
building ⊓ ∀shortThan.¬building — 'a building not shorter than any other building'

Additionally $\mathcal{ALC}$ allows for:

- *TBoxes*, i.e., sets of *axioms*: $C \sqsubseteq D$ 'every C is D'
- *ABoxes*, i.e., sets of *assertions*: $C(a)$ 'a satisfies C'

Extensions of $\mathcal{ALC}$

$\mathcal{ALCO}$ extends $\mathcal{ALC}$ with *nominals*:

$$C := A \mid \{a\} \mid \neg C \mid C \sqcap C \mid C \sqcup C \mid \exists r.\, C \mid \forall r.\, C$$

$\{a\}$ holds only at the individual a

$\mathcal{ALC}_u$ extends $\mathcal{ALC}$ with the *universal role*:

$$C := A \mid \neg C \mid C \sqcap C \mid C \sqcup C \mid \exists r.\, C \mid \forall r.\, C \mid \exists u.\, C \mid \forall u.\, C$$

$\exists u.C$ — 'some individual satisfies C'

Two Types of DDs

Local DDs are of the form $\{\iota C\}$

- $\{\iota(\mathsf{building} \sqcap \forall \mathsf{shortThan}.\neg \mathsf{building})\}$
- expressing *"the tallest building"*

Global DDs are of the form $\iota C.D$

- $\iota(\mathsf{building} \sqcap \forall \mathsf{shortThan}.\neg \mathsf{building}).(\exists \mathsf{locIn}.\mathsf{Dubai})$
- expressing *"the tallest building is located in Dubai"*

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$$(\{\iota C\})^{\mathcal{I}} = \begin{cases} \{d\}, & \text{if } C^{\mathcal{I}} = \{d\}, \\ \emptyset, & \text{otherwise,} \end{cases}$$

Global DDs are of the form $\iota C.D$

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$$(\iota C.D)^{\mathcal{I}} = \begin{cases} \Delta^{\mathcal{I}}, & \text{if } C^{\mathcal{I}} = \{d\} \subseteq D^{\mathcal{I}}, \\ \emptyset, & \text{otherwise.} \end{cases}$$

Description Logics with DDs

$\mathcal{ALC}\iota_L$ — $\mathcal{ALC}$ with local DDs:

$$C := A \mid \neg C \mid C \sqcap C \mid C \sqcup C \mid \exists r.\, C \mid \forall r.\, C \mid \{\iota C\}$$

$\mathcal{ALC}\iota_G$ — $\mathcal{ALC}$ with global DDs:

$$C := A \mid \neg C \mid C \sqcap C \mid C \sqcup C \mid \exists r.\, C \mid \forall r.\, C \mid \iota C.D$$

$\mathcal{ALC}\iota$ — $\mathcal{ALC}$ with local and global DDs:

$$C := A \mid \neg C \mid C \sqcap C \mid C \sqcup C \mid \exists r.\, C \mid \forall r.\, C \mid \{\iota C\} \mid \iota C.D$$

$\mathcal{ALCO}^{\iota}_{u}$ — $\mathcal{ALC}$ with nominals, universal role, and local DDs:

$$C := A \mid \{a\} \mid \neg C \mid C \sqcap C \mid C \sqcup C \mid \exists r.\, C \mid \forall r.\, C \mid \exists u.\, C \mid \forall u.\, C \mid \{\iota C\}$$

Observations

$\iota C.D$ allows to express $\{\iota C\}$: $\quad \{\iota C\} \equiv C \sqcap \iota C.\top$

$\exists u$ and $\{\iota C\}$ allow to express $\iota C.D$: $\quad \iota C.D \equiv \exists u.(\{\iota C\} \sqcap D)$

So in terms of the *expressive power* (equivalence preserving translations):

$$\mathcal{ALC} \preceq \mathcal{ALC}\iota_L \preceq \mathcal{ALC}\iota_G \equiv \mathcal{ALC}\iota \preceq \mathcal{ALCO}_u^{\iota}$$

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Interesting research directions:

- Computational complexity?
- Expressive power characterisation?

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Interesting research directions:

- Computational complexity? **ExpTime-complete**
- Expressive power characterisation? **New bisimulations**

Computational Complexity

Ontology Satisfiability

Theorem. Ontology satisfiability is ExpTime-*complete* for each of

$$\mathcal{ALC}\iota_L, \ \mathcal{ALC}\iota_G, \ \mathcal{ALC}\iota, \text{ and } \mathcal{ALCO}_u^\iota$$

Proof.

- $\mathcal{ALC}$ and $\mathcal{ALCO}_u$ are both ExpTime-complete
- To show that $\mathcal{ALCO}_u^\iota$ is in ExpTime we reduce to $\mathcal{ALCO}_u$
- The reduction translates
  $\{\iota C\}$ into $A_{\iota C} \sqcap \neg \exists u.(A_{\iota C} \land \neg\{a_{\iota C}\})$
  for fresh atomic concept $A_{\iota C}$ and fresh individual name $a_{\iota C}$

$\square$

Concept Satisfiability

Theorem. Concept satisfiability is ExpTime-*complete* for each of
$\mathcal{ALC}\iota_L$, $\mathcal{ALC}\iota_G$, $\mathcal{ALC}\iota$, and $\mathcal{ALCO}^\iota_u$

Proof.

- To show that $\mathcal{ALC}\iota_L$ is ExpTime-hard we reduce satisfiability of an $\mathcal{ALC}$ concept C wrt a TBox $\mathcal{T}$ (which is ExpTime-complete)

- The reduction translates C and $\mathcal{T}$ into

$$C \sqcap \textstyle\bigsqcap_{(D \sqsubseteq E) \in \mathcal{T}} ((\neg D \sqcup E) \sqcap \{\iota\, (\neg(\neg D \sqcup E) \sqcup A_{D \sqsubseteq E})\}),$$

for fresh atomic concept $A_{D \sqsubseteq E}$

$\square$

Expressive Power

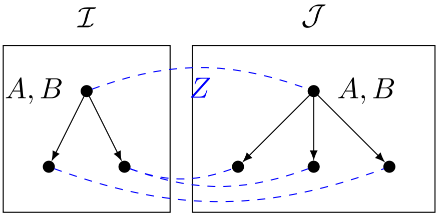
Standard bisimulation...

Definition. An *$\mathcal{ALC}$ bisimulation* is a relation $Z \subseteq \Delta^{\mathcal{I}} \times \Delta^{\mathcal{J}}$ such that for all $(d, e) \in Z$:

Atom: d and e satisfy same atomic concepts

Zig: if $(d, d') \in r^{\mathcal{I}}$, there is e' such that $(e, e') \in r^{\mathcal{J}}$ and $(d', e') \in Z$

Zag: reverse of Zig



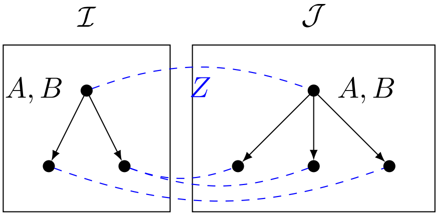
Z is an $\mathcal{ALC}$ bisimulation

Standard bisimulation...

Theorem.

- If d and e are $\mathcal{ALC}$ bisimilar, they satisfy the same $\mathcal{ALC}$ concepts
- If d and e satisfy the same $\mathcal{ALC}$ concepts and $\mathcal{I}, \mathcal{J}$ are ω-saturated, then d and e are $\mathcal{ALC}$ bisimilar

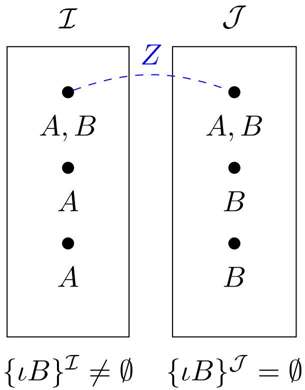
Theorem. An FO formula is invariant under $\mathcal{ALC}$ bisimulations iff it is equivalent to some $\mathcal{ALC}$ concept



Z is an $\mathcal{ALC}$ bisimulation

Standard bisimulation... is inadequate for $\mathcal{ALC}$ with DDs

Z is an $\mathcal{ALC}$ bisimulation, but
does not preserve $\{\iota B\}$ satisfiability.



$\mathcal{ALCO}_u^\iota$ bisimulation

Definition.
Z is an $\mathcal{ALCO}_u^\iota$ *bisimulation* if:

1. Z is an $\mathcal{ALCO}$ bisimualtion
2. Z is total
3. for all $(d, e) \in Z$:
   there is $d' \neq d$ $\mathcal{ALC}$ bisimilar with d iff there is $e' \neq e$ $\mathcal{ALC}$ bisimilar with e such that $(d', e') \in Z$

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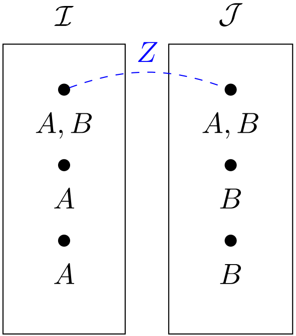

Z is **not an $\mathcal{ALCO}_u^\iota$ bisimulation**
as it does not satisfy conditions 2. and 3.

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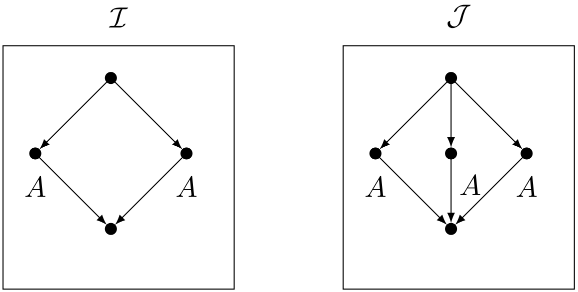

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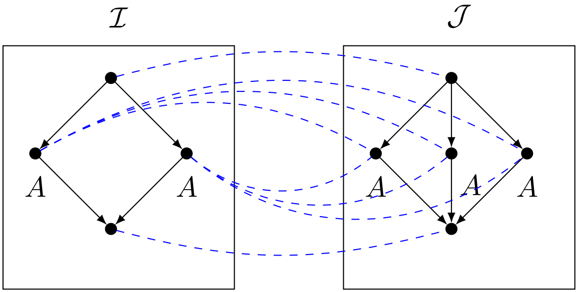

Z is the maximal $\mathcal{ALCO}_u^\iota$ **bisimulation**

$\mathcal{ALCO}_u^\iota$ bisimulation

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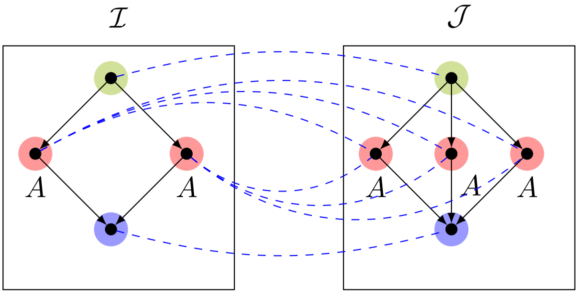

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   there is $d' \neq d$ $\mathcal{ALC}$ bisimilar with d iff there is $e' \neq e$ $\mathcal{ALC}$ bisimilar with e such that $(d', e') \in Z$

Theorem.

- If d and e are $\mathcal{ALCO}_u^\iota$ bisimilar, they satisfy the same $\mathcal{ALCO}_u^\iota$ concepts
- If d and e satisfy the same $\mathcal{ALCO}_u^\iota$ concepts and $\mathcal{I}, \mathcal{J}$ are ω-saturated, then d and e are bisimilar

Theorem. An FO formula is invariant under $\mathcal{ALCO}_u^\iota$ bisimulations iff it is equivalent to some $\mathcal{ALCO}_u^\iota$ concept

$\mathcal{ALC}\iota$ bisimulation

Definition.
Z is an $\mathcal{ALC}\iota$ *bisimulation* if:

1. Z is an $\mathcal{ALC}$ bisimualtion
2. *Names*$(\mathcal{I})$ = *Names*$(\mathcal{J})$

Names$(\mathcal{I}) = \{\mathcal{ALC}$ concept $C \mid$
C holds in a unique individual in $\mathcal{I}\}$

$\mathcal{ALC}\iota$ bisimulation

Definition.
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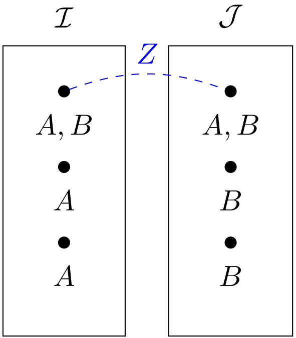

Z is **not an $\mathcal{ALC}\iota$ bisimulation** because *Names*$(\mathcal{I}) \neq$ *Names*$(\mathcal{J})$:

- $B \in$ *Names*$(\mathcal{I})$
- $B \notin$ *Names*$(\mathcal{J})$

$\mathcal{ALC}\iota$ bisimulation

Definition.
Z is an $\mathcal{ALC}\iota$ *bisimulation* if:

1. Z is an $\mathcal{ALC}$ bisimualtion, and
2. *Names*$(\mathcal{I})$ = *Names*$(\mathcal{J})$.

Names$(\mathcal{I}) = \{\mathcal{ALC}$ concept $C \mid$
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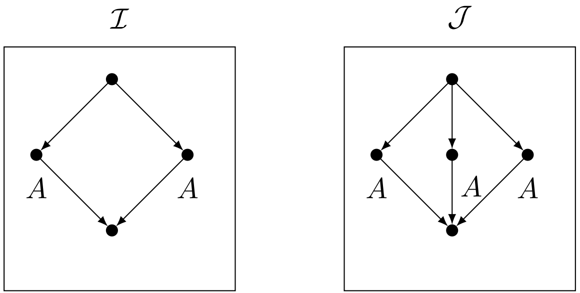
Theorem.

- If d and e are $\mathcal{ALC}\iota$ bisimilar, then they satisfy the same $\mathcal{ALC}\iota$ concepts
- If d and e satisfy the same $\mathcal{ALC}\iota$ concepts and both $\mathcal{I}$ and $\mathcal{J}$ are ω-saturated, then d and e are bisimilar

Theorem. An FO formula is invariant under $\mathcal{ALC}\iota$ bisimulations iff it is equivalent to some $\mathcal{ALC}\iota$ concept

Checking if $Names(\mathcal{I}) = Names(\mathcal{J})$

Ok, but how to check if $Names(\mathcal{J}) = Names(\mathcal{J})$?



$$Names(\mathcal{I}) = \{\exists r.A, \exists r.\exists r.\top, \neg\exists r.\top, \dots\}$$
$$Names(\mathcal{J}) = \{\exists r.A, \exists r.\exists r.\top, \neg\exists r.\top, \dots\}$$

Checking if $Names(\mathcal{I}) = Names(\mathcal{J})$

Ok, but how to check if $Names(\mathcal{J}) = Names(\mathcal{J})$?

Definition.
$\mathit{NamedInd}(\mathcal{I}) = \{d \mid \ d \in C^{\mathcal{I}}$, for some $C \in \mathit{Names}(\mathcal{I})\}$.

Theorem. The following are equivalent (if $\mathit{Names}(\mathcal{I}) \neq \emptyset$ and $\mathit{Names}(\mathcal{J}) \neq \emptyset$):

1. $\mathit{Names}(\mathcal{I}) = \mathit{Names}(\mathcal{J})$,
2. the maximal $\mathcal{ALC}$ bisimulation Z between $\mathcal{I}$ and $\mathcal{J}$ is a total relation and the restriction of Z to $\mathit{NamedInd}(\mathcal{I}) \times \mathit{NamedInd}(\mathcal{J})$ is also total.

Checking if $Names(\mathcal{I}) = Names(\mathcal{J})$

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1. $\mathit{Names}(\mathcal{I}) = \mathit{Names}(\mathcal{J})$,
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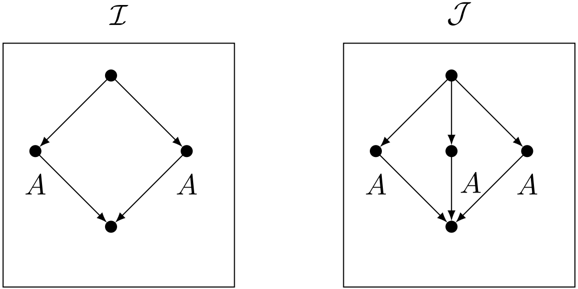

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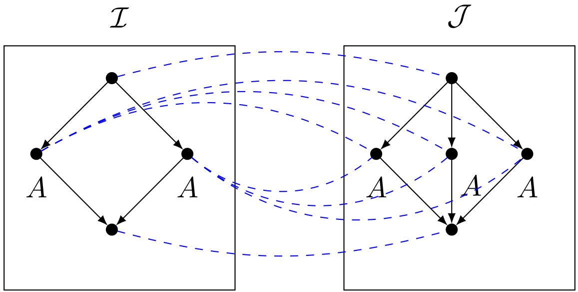

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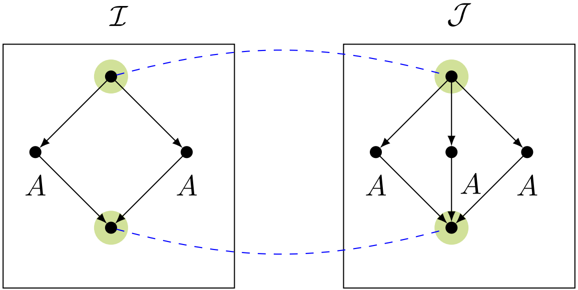


Reasoning in Practice

Tableau Systems

ABox rules:

$$(ABox_{\mathsf{I}})\ \frac{a : C \in ABox}{a : C} \qquad (ABox_r)\ \frac{r(a, a') \in ABox}{r(a, a')}$$

TBox rule:

$$(TBox)\ \frac{C \sqsubseteq D \in TBox, a : E}{a : \neg(C \sqcap \neg D)}$$

Clash rule:

$$(\bot)\ \frac{a : C, a : \neg C}{\bot}$$

Propositional rules:

$$(\neg\neg)\ \frac{a : \neg\neg C}{a : C} \qquad (\sqcap)\ \frac{a : C \sqcap D}{a : C, a : D} \qquad (\neg\sqcap)\ \frac{a : \neg(C \sqcap D)}{a : \neg C \mid a : \neg D}$$

Role rules:

$$(\exists r)\ \frac{a : \exists r.C}{b : C, r : (a, b)} \qquad (\neg\exists r)\ \frac{a : \neg\exists r.C, r : (a, a')}{a' : \neg C}$$

Global definite description rules:

$$(\iota_1^g)\ \frac{a : \iota C.D}{b : C, b : D} \qquad (\iota_2^g)\ \frac{a : \iota C.D, a' : C, a'' : C, a' : E}{a'' : E}$$

$$(\neg\iota^g)\ \frac{a : \neg\iota C.D, a' : E}{a' : \neg C \mid a' : \neg D \mid \begin{array}{c} b : C, b : A_C^g, \\ b' : C, b' : \neg A_C^g \end{array}} \qquad (cut_\iota^g)\ \frac{a : \iota C.D, a' : E}{a' : C \mid a' : \neg C}$$

Local definite description rules:

$$(\iota_1^\ell)\ \frac{a : \{\iota C\}}{a : C} \qquad (\iota_2^\ell)\ \frac{a : \{\iota C\}, a' : C, a'' : C, a' : D}{a'' : D}$$

$$(\neg\iota^\ell)\ \frac{a : \neg\{\iota C\}}{a : \neg C \mid \begin{array}{c} a : \neg A_C, \\ b : C, b : A_C \end{array}} \qquad (cut_\iota^\ell)\ \frac{a : \{\iota C\}, a' : D}{a' : C \mid a' : \neg C}$$

There are *tableau systems for* $\mathcal{ALC}\iota_L$*,* $\mathcal{ALC}\iota_G$*, and* $\mathcal{ALC}\iota$.

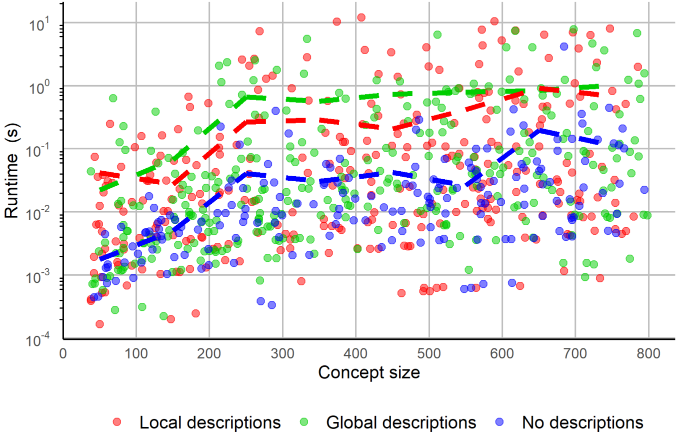
Theorem.
All three systems are
sound, complete, and terminating.

Implementation: OWLeDD Package

Python package: `https://pypi.org/project/OWLeDD/`

```
$ pip install OWLeDD
```

Evaluation:



Takeaways

Definite Descriptions (DDs) introduce new modeling capabilities in description logics

- All $\mathcal{ALC}$ extensions with DDs we consider are ExpTime-*complete* for both concept and ontology satisfiability
- Characterising logics with DDs requires a *new type of bisimulations*
- Local DDs $\{\iota C\}$ are *strictly less expressive* than global DDs $\iota C.D$
- *In practice* global DDs have *higher computational cost*

Future work:

- Lower complexity logics with DDs, e.g. $\mathcal{EL}$ family
- Complexity-optimal tableau systems
- Applications, e.g. computing referring expressions

Thank you for your attention!

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