Definite Descriptions in First-Order Modal and Temporal Logics

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Definite Descriptions in KR Languages - DefDesKR
Half-Day Tutorial @ KR 2026

First-Order Modal Logics with Non-Rigid Designators and Counting

First-Order Modal and Temporal Logics

Extensions of First-Order Logic (FOL) with modal/temporal operators

- Full logics (highly) **undecidable**: contain FOL
- Quest for **decidable fragments**: generalisation of modal/temporal DLs

Non-Rigid Designators and Counting (NRDC) Features

- **equality/counting**;
- **non-rigid** and possibly **non-denoting constants**
- **definite descriptions**

First-Order Modal Logic with DDs: Background and Challenges

The Bad

Modal extensions of **decidable FO fragments** are typically **undecidable**, e.g.:

- Monadic fragment of FO **decidable**
- Monadic fragment of FOMLs $\mathbf{K}_n$ and $\mathbf{S5}_n$, with $n \geq 1$, **undecidable**

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Monodic fragments: modalities applied only to formulas with ≤ 1 **free variable**

- often preserve decidability of underlying FO fragments…
- …but rely on the absence of NRDC features!

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The Ugly

Mostly **negative results** on computational behaviour of fragments with NRDC features

- from product modal logics & fragments of FO modal/temporal logics with counting

First-Order Modal Logic with DDs: Recent Results

Investigation of **decidability** and **complexity boundaries**
for **monodic fragments** with **NRDC features**

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- **Decision problems**: **validity** and **global consequence**
  - Global consequence **not reducible** to validity in general, but...
  - ...**reducible** on frames w/: single equivalence relation; transitive closure; or time flows

$Q^{=}ML_\iota$, the First-Order Modal Language with NRDC

Terms and Formulas

$Q^{=}ML_\iota$ **terms** and **formulas** defined by mutual induction:

$$\tau ::= x \mid c \mid \iota x.\varphi$$
$$\varphi ::= P(\tau_1, \ldots, \tau_m) \mid \tau_1 = \tau_2 \mid \neg\varphi \mid (\varphi_1 \wedge \varphi_2) \mid \exists x\, \varphi \mid \Diamond_a \varphi$$

- **variables** $x \in \mathrm{Var}$, **constants** $c \in \mathrm{Con}$, and **predicates** $P \in \mathrm{Pred}$ (m-ary)
- finite set of **modalities** $a \in A$
- **definite descriptions** $\iota x.\varphi$, read as "the x such that φ"

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Notation

- $\varphi_1 \vee \varphi_2 := \neg(\neg\varphi_1 \wedge \neg\varphi_2)$, $\varphi_1 \to \varphi_2 := \neg\varphi_1 \vee \varphi_2$, $\varphi_1 \leftrightarrow \varphi_2 := (\varphi_1 \to \varphi_2) \wedge (\varphi_2 \to \varphi_1)$
- $\forall x\, \varphi := \neg\exists x\, \neg\varphi$
- $\Box_a \varphi := \neg\Diamond_a \neg\varphi$
- Γ finite set of sentences (no free variables)

Semantics with Partial Interpretations

Partial Interpretation with Expanding Domains

$\mathfrak{M} = (\mathfrak{F}, \Delta, \cdot)$ where:

- $\mathfrak{F} = (W, \{R_a\}_{a \in A})$ **frame** with worlds W ($\neq \emptyset$) and accessibility relations R_a
- Δ function assigning **domain** Δ_w ($\neq \emptyset$) to each $w \in W$ s.t. $\Delta_w \subseteq \Delta_v$ when $wR_a v$
- $\cdot$ function mapping each $w \in W$ to **partial FO interpretation** $\mathfrak{M}(w)$ with:
  - $P^{\mathfrak{M}(w)} \subseteq \Delta_w^m$, for each m-ary predicate $P \in \mathrm{Pred}$ (total on predicates)
  - $c^{\mathfrak{M}(w)} \in \Delta_w$, for **some constant** symbols $c \in \mathrm{Con}$ (**partial** on constants)

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Designation, Total Interpretations, Constant Domains, Assignments

- c **designates at** w: $c^{\mathfrak{M}(w)}$ is defined
- **total interpretation**: all constants designate at all worlds
- **constant domains**: $\Delta_w = \Delta_v$ for all $w, v \in W$
- **assignment at** w: function $\mathfrak{a}$ from Var to Δ_w
- x-**variant** of $\mathfrak{a}$ at w: assignment $\mathfrak{a}'$ at w that can differ from $\mathfrak{a}$ only on x

Term Interpretation and Satisfaction

Value of Terms

$$\tau^{\mathfrak{M}(w),\mathfrak{a}} = \begin{cases} \mathfrak{a}(x), & \text{if } \tau = x \in \mathsf{Var} \\ c^{\mathfrak{M}(w)}, & \text{if } \tau = c \in \mathsf{Con} \text{ and } c^{\mathfrak{M}(w)} \textbf{ defined} \\ \mathfrak{a}'(x), & \text{if } \tau = \iota x.\varphi \text{ and } \mathfrak{M}, w \models^{\mathfrak{a}'} \varphi \text{ for } \textbf{exactly one } x\text{-variant } \mathfrak{a}' \text{ of } \mathfrak{a} \\ \textbf{undefined}, & \text{otherwise} \end{cases}$$

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Satisfaction Relation

- $\mathfrak{M}, w \models^{\mathfrak{a}} P(\tau_1, \ldots, \tau_m)$ iff all $\tau_i^{\mathfrak{M}(w),\mathfrak{a}}$ **defined** and $(\tau_1^{\mathfrak{M}(w),\mathfrak{a}}, \ldots, \tau_m^{\mathfrak{M}(w),\mathfrak{a}}) \in P^{\mathfrak{M}(w)}$
- $\mathfrak{M}, w \models^{\mathfrak{a}} \tau_1 = \tau_2$ iff both $\tau_i^{\mathfrak{M}(w),\mathfrak{a}}$ **defined** and $\tau_1^{\mathfrak{M}(w),\mathfrak{a}} = \tau_2^{\mathfrak{M}(w),\mathfrak{a}}$
- $\mathfrak{M}, w \models^{\mathfrak{a}} \exists x\, \varphi$ iff there exists x-variant $\mathfrak{a}'$ with $\mathfrak{M}, w \models^{\mathfrak{a}'} \varphi$
- $\mathfrak{M}, w \models^{\mathfrak{a}} \Diamond_a \varphi$ iff there exists $v \in W$ such that $wR_a v$ and $\mathfrak{M}, v \models^{\mathfrak{a}} \varphi$

Truth, Validity, and Global Consequence

Truth, Satisfaction

- φ **true in** $\mathfrak{M}$, $\mathfrak{M} \models \varphi$: φ satisfied under every assignment at every world of $\mathfrak{M}$
- φ **satisfied in** $\mathfrak{M}$: φ satisfied under some assignment at some world of $\mathfrak{M}$
- Γ **true in** $\mathfrak{M}$, $\mathfrak{M} \models \Gamma$: every sentence in Γ is true in $\mathfrak{M}$

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Validity, Satisfiability, Global Consequence

$\mathcal{C}$ class of frames (see below). Formula φ

- $\mathcal{C}$-**valid**: φ true in every interpretation $\mathfrak{M}$ based on a frame $\mathfrak{F} \in \mathcal{C}$
- $\mathcal{C}$-**satisfiable**: φ satisfied in some interpretation $\mathfrak{M}$ based on a frame $\mathfrak{F} \in \mathcal{C}$
- **global $\mathcal{C}$-consequence of theory** Γ: φ true in any interpretation $\mathfrak{M}$ based on a frame in $\mathcal{C}$ such that $\mathfrak{M} \models \Gamma$

Classes of Frames for Modal and Temporal Logics

$\mathbf{K}_n$ and $\mathbf{S5}_n$ Frames

- $\mathbf{K}_n$: class of all frames with n **arbitrary** accessibility relations
- $\mathbf{S5}_n$: class of frames with n **equivalence relations**; $\mathrm{S5} := \mathrm{S5}_1$

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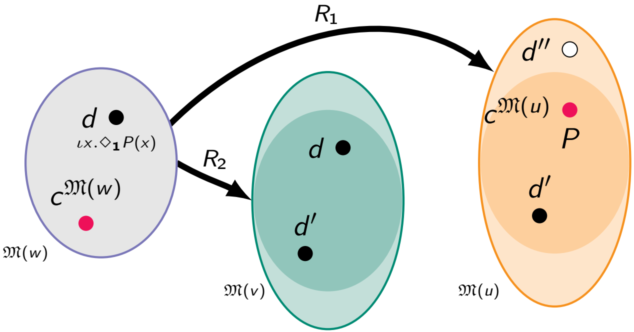
Linear Temporal Frames

- $\mathbf{LTL}^{\diamond}$: $\{(\mathbb{N}, <)\}$, with standard **strict linear order** $<$ (interpreting $\diamond$)
- $\mathbf{LTL}\boldsymbol{f}^{\diamond}$: $\{\,(\{0, \ldots, n\}, <) \mid n \in \mathbb{N}\,\}$, with $<$ restricted to **finite** $\{0, \ldots, n\}$
- **LTL**: $\{(\mathbb{N}, <, S)\}$, with **successor** relation $S = \{(i, i+1) \mid i \in \mathbb{N}$ (interpreting $\bigcirc$)
- $\mathbf{LTL}\boldsymbol{f}$: $\{\,(\{0, \ldots, n\}, <, S) \mid n \in \mathbb{N}\,\}$, with $<$ and S restricted to **finite** $\{0, \ldots, n\}$

Examples

Partial Interpretation with Non-Rigid Designators



$$\exists x(x \neq c \wedge \Diamond_1(x = c))$$

unsatisfiable if **constants** are **rigid**

Examples

Vulcan and Venus

- "It is conceivable *that* Vulcan is the planet orbiting between Sun and Mercury":

$$\Diamond(\mathsf{vulcan} = \iota z.\mathsf{OrbitsBetween}(z, \mathsf{sun}, \mathsf{mercury}))$$

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- “Even though such a planet does not exist”:

$$\neg\exists x(x = \mathrm{vulcan}) \land \neg\exists x(x = \iota z.\mathrm{OrbitsBetween}(z, \mathrm{sun}, \mathrm{mercury}))$$

i.e., neither ‘vulcan’ nor ‘$\iota z.\mathrm{OrbitsBetween}(z, \mathrm{sun}, \mathrm{mercury})$’ designate in this world

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- "It is known *of* the planet orbiting between Mercury and Earth that it is Venus":

$$\exists x(x = \iota z.\mathsf{OrbitsBetween}(z, \mathsf{mercury}, \mathsf{earth}) \wedge \Box(x = \mathsf{venus}))$$

  in **S5** frames, this implies that 'venus' is rigid

Monodic Fragments

Monodic Fragment

Set $\mathsf{Q}^{=}_{\boxed{1}}\mathsf{ML}_{\iota}$ of **monodic** formulas:

- every subformula of the form $\Diamond_a \psi$ has **at most one free variable**.

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Minimal Sub-Fragment

- $Q^{1=}ML_\iota$: **One-variable** fragment with $\leq$ 1-ary predicates (and equality)

Maximal Sub-Fragments

- $C^2_{\boxed{1}}ML_\iota$: **Two-variable** fragment with **counting** quantifiers and $\leq$ 2-ary predicates
- $GF^{=}_{\boxed{1}}ML_\iota$: **Guarded** fragment with equality

One-Variable Fragment $Q^{1=}ML_\iota$

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$Q^{1=}ML_\iota$ **terms** and **formulas** built with:

- **one variable** only
- predicates of **arity at most one** (plus equality)
- constants and definite descriptions

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Remarks

- Underpinned by FO^1 with equality and constants
- All formulas trivially monodic
- Variants extensively studied as product modal logics

Two-Variable Fragment with Counting $C^2_{\boxed{1}}ML_\iota$

$C^2_{\boxed{1}}ML_\iota$ **terms** and **formulas** built with:

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- **counting quantifiers** $\exists^{\geq k}x$, $k \geq 0$
- predicates of **arity at most two** (including equality)
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Counting Quantifier

$$\mathfrak{M}, w \models^{\mathfrak{a}} \exists^{\geq k} x\, \varphi \quad \text{iff} \quad \mathfrak{M}, w \models^{\mathfrak{a}'} \varphi \text{ for at least } k \text{ distinct } x\text{-variants } \mathfrak{a}'$$

Guarded Fragment $\mathrm{GF}^{=}_{\boxed{1}}\mathrm{ML}_\iota$

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Remark

Closed definite descriptions necessary for decidability even without modalities

- $\forall x F(x, \iota y.F(x, y))$ ensures F is a **function**
- guarded fragment with **functionality** is **undecidable**

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$\exists x\, \varphi(x)$ can still be expressed

- equivalent to $\exists x\, ((x = x) \wedge \varphi(x))$, with $x = x$ as its guard.

Decision Problems

Validity and Global Consequence Decision Problems

For fragment $\mathcal{L}$ and frame class $\mathcal{C}$

- $\mathcal{C}$**-validity in** $\mathcal{L}$: Is φ true on all interpretations based on $\mathcal{C}$-frames?
- **global** $\mathcal{C}$**-consequence in** $\mathcal{L}$: Is φ true in all interpretations based on $\mathcal{C}$-frames that make theory Γ true?

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Problem Qualifiers

- **total $\mathcal{C}$-validity**: restriction to total interpretations
- with **constant** domains, with **expanding** domains

Overview of Results

frames $\mathcal{C}$	dom.	$\mathcal{C}$-validity			global $\mathcal{C}$-consequence		
		$Q^{1=}ML_\iota$	$C^2_{\boxed{1}}ML_\iota$	$GF^{=}_{\boxed{1}}ML_\iota$	$Q^{1=}ML_\iota$	$C^2_{\boxed{1}}ML_\iota$	$GF^{=}_{\boxed{1}}ML_\iota$
S5	$=$	coNExp	coNExp	2Exp	coNExp	coNExp	2Exp
$\mathbf{S5}_n$, $n \geq 2$	$=$	coNExp	coNExp	2Exp	undecidable		
$\mathbf{K}_n$	$=$	coNExp	coNExp	2Exp	undecidable		
	$\subseteq$	PSpace	coNExp	2Exp	?		
$\mathbf{K}_{*n}$, $\mathbf{LTL}^{(\Diamond)}$	$=$	Σ^1_1					
	$\subseteq$	undecidable					
$\mathbf{K}f_{*n}$, $\mathbf{LTL}f^{(\Diamond)}$	$=$	undecidable					
	$\subseteq$	decidable, Ackermann-hard					

Preliminary Observations

Elsewhere Operator

- Reductions between our one-variable fragment with **non-rigid constants** and $\mathrm{Q}^{1\neq}\mathrm{ML}$, the one-variable fragment with **difference/elsewhere** quantifier $\exists^{\neq} x$:

$$\varphi ::= \; P(x) \mid \neg\varphi \mid (\varphi \wedge \varphi) \mid \; \exists x\, \varphi \mid \boxed{\exists^{\neq} x\, \varphi} \mid \Diamond_a \varphi$$

$$\mathfrak{M}, w \models^{\mathfrak{a}} \boxed{\exists^{\neq} x\, \varphi} \quad \text{iff} \quad \mathfrak{M}, w \models^{\mathfrak{a}'} \varphi, \; \text{for some } x\text{-variant } \boxed{\mathfrak{a}' \neq \mathfrak{a}} \text{ at } w$$

[Hampson & Kurucz, AiML 2012; Hampson & Kurucz, ACM TOCL 2015]

- Lift results from propositional case: "**difference** $\equiv$ **nominals** + **universal** modality"
  [Gargov & Goranko, JPL, 1993]
  - from $\mathrm{Q}^{1=}\mathrm{ML}_c$ to $\mathrm{Q}^{1\neq}\mathrm{ML}$: $x = c \;\rightsquigarrow\; Q_c(x) \wedge \neg\exists^{\neq} x\, Q_c(x)$
  - from $\mathrm{Q}^{1\neq}\mathrm{ML}$ to $\mathrm{Q}^{1=}\mathrm{ML}_c$: $\exists^{\neq} x\, \psi \;\rightsquigarrow\; \exists x\, \psi(x) \wedge \big(x = c_\psi \rightarrow \exists x\, (\neg(x = c_\psi) \wedge \psi(x))\big)$

Preliminary Observations

Getting Rid of Definite Descriptions...

Normalise and **eliminate definite descriptions**: replace definite descriptions $\iota x.\psi(x, \boldsymbol{y})$ with "atomic surrogates" $\iota x.P_{\psi}(x, \boldsymbol{y})$; then, replace atoms $\alpha(\iota x.Q(x, \boldsymbol{y}), \boldsymbol{\tau})$ with their "**Russell's paraphrase**":

- in $\mathsf{Q}^{=}_{\boxed{1}}\mathsf{ML}_{\iota}$, $\exists x(\alpha(x, \boldsymbol{\tau}) \wedge Q(x, \boldsymbol{y}) \wedge \forall x'(Q(x', \boldsymbol{y}) \rightarrow x' = x))$
- in $\mathsf{C}^{2}_{\boxed{1}}\mathsf{ML}_{\iota}$ $\exists x(\alpha(x, \boldsymbol{\tau}) \wedge Q(x, \boldsymbol{y})) \wedge \exists^{=1} x Q(x, \boldsymbol{y})$
- in $\mathsf{Q}^{1=}\mathsf{ML}_{\iota}$ and $\mathsf{GF}^{=}_{\boxed{1}}\mathsf{ML}_{\iota}$ $\alpha(c_{\iota x.Q(x)}, \boldsymbol{\tau}) \wedge Q(c_{\iota x.Q(x)}) \wedge \forall x \left(Q(x) \rightarrow x = c_{\iota x.Q(x)}\right)$

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- in $\mathsf{C}^{2}_{\boxed{1}}\mathsf{ML}_{\iota}$ $\exists x(\alpha(x, \boldsymbol{\tau}) \land Q(x, \boldsymbol{y})) \land \exists^{=1} x Q(x, \boldsymbol{y})$
- in $\mathsf{Q}^{1=}\mathsf{ML}_{\iota}$ and $\mathsf{GF}^{=}_{\boxed{1}}\mathsf{ML}_{\iota}$ $\alpha(c_{\iota x.Q(x)}, \boldsymbol{\tau}) \land Q(c_{\iota x.Q(x)}) \land \forall x \left(Q(x) \to x = c_{\iota x.Q(x)}\right)$

...And Other Useful Reductions

- **Reduce partial** to **total** interpretations, and **viceversa**:
  - Partial $\rightsquigarrow$ Total: replace constants with propositional letter designation switch
$$P(\tau_1, \dots, \tau_m) \quad \rightsquigarrow \quad \bigwedge_{c_i \in \{\boldsymbol{\tau_1}, \dots, \tau_m\}} p_{c_i} \land P(\tau_1, \dots, \tau_m)$$
  - Total $\rightsquigarrow$ Partial: add "**designation axioms**" for $c \rightsquigarrow \exists x(x = c)$
- **Reduce expanding** to **constant** domains

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$\mathbf{K}_n$	$=$	coNExp	coNExp	2Exp	undecidable		
	$\subseteq$	PSpace	coNExp	2Exp	?		
$\mathbf{K}_{*n}$, $\mathbf{LTL}^{(\Diamond)}$	$=$	Σ^1_1					
	$\subseteq$	undecidable					
$\mathbf{K}f_{*n}$, $\mathbf{LTL}f^{(\Diamond)}$	$=$	undecidable					
	$\subseteq$	decidable, Ackermann-hard					

Main Ideas for Decidability

From Models to Quasimodels

Adapt **quasimodel** machinery for NRDC features

- Use **multisets** of types and runs to handle counting
- ...but they can still require **infinite branching**

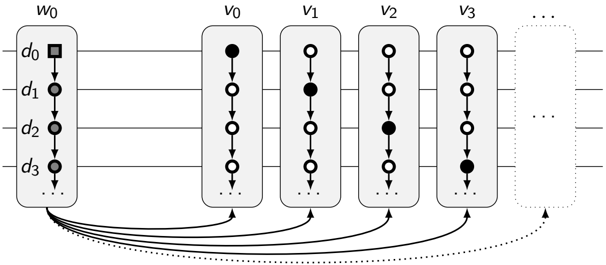

Figure: $\forall x \exists^{=1} y\, P(x, y) \wedge \forall x \exists^{\leq 1} z\, P(z, x) \wedge \exists x \neg \exists z\, P(z, x) \wedge \forall x \Diamond_a A(x) \wedge \Box_a \exists^{\leq 1} x\, A(x)$

Main Ideas for Decidability

From Quasimodels to Weak Quasimodels

Introduce **weak quasimodels** with **relaxed saturation conditions**

- Saturate only a representative for each multiset of types
- Avoid infinite branching of quasimodels due to counting

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Bounded-Size Weak Quasimodels

Use **bounded-size** weak quasimodels to decide satisfiability

- 1-variable and guarded fragments: use weak quasimodels directly
- 2-variable with counting: introduce weak pre-quasimodels + (in)equalities encoding constraints in Presburger Arithmetic with infinity

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Simpler Expanding Domain Cases

- validity/global consequence in fragments $\begin{cases} \text{with transitive closure \& no infinite chain} \\ \text{on finite temporal frames} \end{cases}$
- $\mathbf{K}_n$-validity in one-variable fragment

Overview of Undecidability Results

frames $\mathcal{C}$	dom.	$\mathcal{C}$-validity			global $\mathcal{C}$-consequence		
		$Q^{1=}ML_\iota$	$C^2_{\boxed{1}}ML_\iota$	$GF^{=}_{\boxed{1}}ML_\iota$	$Q^{1=}ML_\iota$	$C^2_{\boxed{1}}ML_\iota$	$GF^{=}_{\boxed{1}}ML_\iota$
S5	$=$	coNExp	coNExp	2Exp	coNExp	coNExp	2Exp
$\mathbf{S5}_n$, $n \geq 2$	$=$	coNExp	coNExp	2Exp	undecidable		
$\mathbf{K}_n$	$=$	coNExp	coNExp	2Exp	undecidable		
	$\subseteq$	PSpace	coNExp	2Exp	?		
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	$\subseteq$	decidable, Ackermann-hard					

Main Ideas for Undecidability

Main Ideas for Undecidability

- For global consequence on $\mathbf{K}_n$ ($n \geq 1$) and $\mathbf{S5}_n$ ($n \geq 2$) frames, reduce:
  - **undecidable** one-variable $\mathbf{K}_n$ fragment with **elsewhere quantifier** $\exists^{\neq} x$
  - to our one-variable fragment with **non-rigid constants**
- For validity/global consequence with **transitive closure** & on **temporal** (infinite with constant/expanding domains, or finite with constant domains) frames, reduce
  - **undecidable one-variable FOTL with elsewhere quantifier** to our one-variable fragments on temporal (infinite or finite, resp.) frames
  - the latter to one-variable fragments with **transitive closure** (with or without infinite chains, resp.)

Other Approaches to First-Order Modal Logic with Non-Rigid Terms

An Ocean of Possibilities

- Syntax: scope disambiguation for non-rigid terms and modal operators with predicate λ-abstraction / typed languages; DDs as referring expressions for time points: "the first time φ was true"
- Proof theory: axiom systems, tableaux, sequent calculi
- Semantics: double-domain, counterparts, modal categories, sheaves, . . .

[Garson, *HPL*, 2001; Corsi, JSL 2002; Shehtman, AiML 2006, Braüner & Ghilardi, *HML*, 2007; Awodey & Kishida, AiML 2012; Indrzejczak, AiML 2020; Fitting & Mendelsohn, *FOML*, 2023; Indrzejczak & Zawidzki, *Synthese* 2023; Orlandelli, JPL 2024, Shkatov & Shehtman, ESSLLI 2025; Ghilardi & Marquès, JSL 2025]